

THE EXISTENCE AND BOUNDEDNESS OF MULTILINEAR MARCINKIEWICZ INTEGRALS ON COMPANATO SPACES

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ABSTRACT. In this paper, we established the boundedness of m -linear Marcinkiewicz integral on Campanato type spaces. We showed that if the m -linear Marcinkiewicz integral is finite for one point, then it is finite almost everywhere. Moreover, the following norm inequality holds,

$$\|\mu(\vec{f})\|_{\mathcal{E}^{\alpha,p}} \leq C \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,p_j}},$$

where $\mathcal{E}^{\alpha,p}$ is the classical Campanato spaces.

1. INTRODUCTION AND MAIN RESULTS

It is well known that the following classical Marcinkiewicz integral of higher dimension was first introduced and studied by E. M. Stein [27] in 1958.

$$(1.1) \quad \mu_{\Omega}(f)(x) = \left(\int_0^{\infty} \left| \int_{|y| \leq t} f(x-y) \frac{\Omega(y)}{|y|^{n-1}} dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

Stein showed that the Marcinkiewicz integral μ_{Ω} is of weak type $(1, 1)$ and type (p, p) ($1 < p \leq 2$), where the Lipschitz continuous function Ω is homogeneous of degree zero and its integration on the unit sphere vanishes. Later, Stein's L^p result was further extended by Benedek, Calderón and Panzone [1] to the case $1 < p < \infty$ when the kernel belongs to $C^1(S^{n-1})$. In the connection of μ_{Ω} , the parametric Marcinkiewicz integral operator $\mu_{\Omega,p}$ was first considered by Hörmander [19] in 1960. Since then, many works have been done for Marcinkiewicz integral or its related parametric operators. L^p boundedness for these operators were well discussed [10, 11, 12] and there were also some related results on function spaces, such as Triebel-Lizorkin spaces $\dot{F}_{pq}^{\alpha}(\mathbb{R}^n)$ [2, 3, 36], Hardy spaces [14, 33], Campanato spaces, [29, 30, 31, 13]. A nice survey was given by Lu [24].

It is also well known that, the multilinear operators were first introduced and studied by Coifman and Meyer [7, 8] in the 70s. After the celebrated works of them, the topic was retaken by several authors, including Christ and Journé [5], Kenig and Stein

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[21], Grafakos and Torres [15, 16] and Lerner et al [23]. The study of multilinearization of Littlewood-Paley's square function can be traced back to the work of Coifman and Meyer [9]. Some improvement can be found in the works of Yabuta [38], Sato and Yabuta [32]. Recently, some weighted results for multilinear Littlewood-Paley operators, in particular, the multilinear Marcinkiewicz integral, were established in [4]. To state some known results, we first introduce some definitions.

Definition 1.1. (Multilinear Marcinkiewicz integral [4]). Let Ω be a function defined on $(\mathbb{R}^n)^m$ with the following properties: Let Ω be a function defined on $(\mathbb{R}^n)^m$ with the following properties:

(i) Ω is homogeneous of degree 0, i.e.,

$$(1.2) \quad \Omega(\lambda \vec{y}) = \Omega(\vec{y}); \quad \text{for any } \lambda > 0 \text{ and } \vec{y} = (y_1, \dots, y_m) \in (\mathbb{R}^n)^m.$$

(ii) Ω is Lipschitz continuous on $(\mathbb{S}^{n-1})^m$, i.e. there are $0 < \alpha < 1$ and $C > 0$ such that for any $\xi = (\xi_1, \dots, \xi_m), \eta = (\eta_1, \dots, \eta_m) \in (\mathbb{R}^n)^m$

$$(1.3) \quad |\Omega(\xi) - \Omega(\eta)| \leq C |\xi' - \eta'|^\alpha,$$

where $(y_1, \dots, y_m)' = \frac{(y_1, \dots, y_m)}{|y_1| + \dots + |y_m|}$, and it should be noted that $(y_1, \dots, y_m)'$ is not an element of $(\mathbb{S}^{n-1})^m$;

(iii) The integration of Ω on each unit sphere vanishes,

$$(1.4) \quad \int_{\mathbb{S}^{n-1}} \Omega(y_1, \dots, y_m) dy_j = 0, \quad j = 1, \dots, m.$$

For any $\vec{f} = (f_1, \dots, f_m) \in S \times \dots \times S$, we can define the operator F_t for any $t > 0$ as

$$(1.5) \quad \begin{aligned} F_t(\vec{f})(x) &= \frac{\chi_{(B(0,t))^m} \Omega(\vec{\cdot})}{t^m |\vec{\cdot}|^{m(n-1)}} * (f_1 \otimes \dots \otimes f_m)(x) \\ &= \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y}, \end{aligned}$$

where $|\vec{y}| = |y_1| + \dots + |y_m|$ and $B(x, t) = \{y \in \mathbb{R}^n : |y - x| \leq t\}$. Finally, the multilinear Marcinkiewicz integral μ is defined by

$$(1.6) \quad \mu(\vec{f})(x) = \left(\int_0^\infty |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

If $m = 1$, it is easy to see that $\mu(\vec{f})$ coincides with $\mu_\Omega(f)$ which was defined and studied by Stein [27]. If $m \geq 2$, Chen, Xue and Yabuta [4] recently gave the following result.

Theorem A (Estimate for μ) ([4]). Suppose μ is bounded from $L^{q_1} \times \dots \times L^{q_m}$ to L^q for some $1 < q_1, \dots, q_m < \infty$ with $1/q = 1/q_1 + \dots + 1/q_m$. Then for $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, there is a $C > 0$ such that

$$\|\mu(\vec{f})\|_{L^p} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}}.$$

To state some other results, we begin with the definition of Campanato type spaces.

Definition 1.2. (Campanato space). ([13]) Let $1 \leq p < \infty$ and $-n/p \leq \alpha < 1$. A locally integrable function f is said to belong to the Campanato space $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for any ball $B \subset \mathbb{R}^n$,

$$(1.7) \quad \|f\|_{\mathcal{E}^{\alpha,p}} := \sup_B \frac{1}{|B|^{\alpha/n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{1/p} < \infty,$$

where the supremum is taken over all cubes in \mathbb{R}^n with sides parallel to the coordinate axes, and f_B denotes the average of f over B , that is, $f_B = \frac{1}{|B|} \int_B f(x) dx$.

Remark 1.3. It is well known that if $\alpha \in (0, 1)$ and $p \in [1, \infty)$, then $\mathcal{E}^{\alpha,p}(\mathbb{R}^n) = Lip_\alpha(\mathbb{R}^n)$, with equivalent norms; if $\alpha = 0$, then $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ coincides with $BMO(\mathbb{R}^n)$; and if $\alpha \in (-n/p, 0)$, then $\mathcal{E}^{\alpha,p}(\mathbb{R}^n)$ coincides with the Morrey space $L^{p,\alpha p+n}(\mathbb{R}^n)$.

In 1987, Han [17] proved the following result, which was improved later by Lu, Ding and Xue [13] with more weaker conditions assumed on the kernel.

Theorem B. *Suppose Ω is continuous on \mathcal{S}^{n-1} , satisfies a Lip_α condition for $0 < \alpha \leq 1$, and its integration on \mathcal{S}^{n-1} vanishes. If $f \in BMO(\mathbb{R}^n)$ and $\mu_\Omega(f)(x)$ is finite on a set of positive measure, then $\mu_\Omega(f)(x)$ is finite a.e. on \mathbb{R}^n , and there exists a positive constant C , independent of f , such that*

$$\|\mu_\Omega(f)\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{BMO(\mathbb{R}^n)}.$$

Remark 1.4. Previous works can be traced back to the celebrated theorem given by Wang [29] and Kurtz [22] for g -function. Theorem A also holds for Lusin's Area integral, Littlewood-Paley g_λ^* -function. Moreover, similar results were treated for the above operators on Campanato type spaces ([25], [26], [13], [18]).

In 1990, Wang and Chen [31] gave the following interesting result.

Theorem C. *If $f \in BMO(\mathbb{R}^n)$ and $\mu_\Omega(f)(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu_\Omega(f)(x)$ is finite a.e. on \mathbb{R}^n , and there exists a positive constant C , independent of f , such that $\|\mu_\Omega(f)\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{BMO(\mathbb{R}^n)}$.*

Similarly results still hold for Littlewood-Paley g -function, Lusin's area integral S , Littlewood-Paley g_λ^* -function.

Remark 1.5. In 2003, Yabuta [39] extended Theorem C to the case of Campanato type spaces, also for g_λ^* -function and Marcinkiewicz integral. In addition, in 2004, Sun [34] got similar results for g -function and Lusin's area integral.

In this paper, we shall give a multilinear analogue of Theorem C for the multilinear Marcinkiewicz integral. Our main results are as follows.

Theorem 1.1. *Let Ω be a function defined on \mathbb{R}^{mn} , satisfying (1.2), Lipschitz continuous condition (1.3) and (1.4). Suppose that μ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_1, \dots, q_m < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$. For $f_i \in BMO(\mathbb{R}^n)$, if*

$\mu(\vec{f})(x)$ is finite for a point $x_0 \in \mathbb{R}^n$, then $\mu(\vec{f})(x) < \infty$ a.e. on \mathbb{R}^n , and there exists a positive constant C , independent of \vec{f} , such that

$$\|\mu(\vec{f})\|_{BMO} \leq C \prod_{i=1}^m \|f_i\|_{BMO}.$$

Theorem 1.2. *Let Ω be a function defined on \mathbb{R}^{mn} , satisfying (1.2), Lipschitz continuous condition (1.3) and (1.4) with index replaced by β . Suppose that μ is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q_1, \dots, q_m < \infty$ with $1/q = 1/q_1 + \cdots + 1/q_m$. Suppose that $-\infty < \alpha = \alpha_1 + \cdots + \alpha_m < 0$ with $\alpha_1, \dots, \alpha_m < 0$ and $n < p < \infty$, or $-m < \alpha < 0$ and $1 < p < \infty$, then for $f_j \in \mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)$, and $1 < p_1, \dots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\mu(\vec{f})$ is either infinite everywhere or finite almost everywhere, and in the latter case, the following inequality holds,*

$$\|\mu(\vec{f})\|_{\mathcal{E}^{\alpha, p}} \leq C \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}}.$$

Theorem 1.3. *Let Ω be the same as in the above theorem. Suppose that $0 < \alpha < \beta/2$, and $f_j \in Lip_{\alpha_j}$, $j = 1, \dots, m$. Then $\mu(\vec{f})$ is either infinite everywhere or finite almost everywhere and in the latter case,*

$$\|\mu(\vec{f})\|_{Lip_{\alpha}} \leq C \prod_{j=1}^m \|f_j\|_{Lip_{\alpha_j}},$$

where $C > 0$ is independent of \vec{f} .

The article is organized as follows. Some basic lemmas will be presented in Section 2. The proof of Theorem 1.1 will be given in Section 3. In Section 4, we will demonstrate the proofs of Theorem 1.2 and Theorem 1.3. Finally, in Section 5, some extensions will be presented for the operators with separated kernels and more general type kernels.

2. SOME BASIC LEMMAS

To prove our Theorems, we prepare some lemmas.

Lemma 2.1. *Let $1 \leq p < \infty$, $B = B(x_0, r)$, $x \in B$ and $t > 8r > 0$. Then for $0 \leq k \leq k_0$ with $k_0 \in \mathbb{N}$ satisfying $2r \leq 2^{-k_0}t < 4r$ we have*

$$\begin{aligned} \left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \right)^{\frac{1}{p}} &\leq c_n \left(\sum_{j=k}^{k_0} 2^{-j\alpha} \right) t^{\alpha} \|f\|_{\mathcal{E}^{p, \alpha}} \\ &\leq \begin{cases} cr^{\alpha} \|f\|_{\mathcal{E}^{p, \alpha}} & \alpha < 0, \\ c \log \frac{t}{r} \|f\|_{\mathcal{E}^{p, \alpha}} & \alpha = 0, \\ ct^{\alpha} \|f\|_{\mathcal{E}^{p, \alpha}} & \alpha > 0, \end{cases} \end{aligned}$$

Proof. We see easily that

$$\begin{aligned} & \left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \right)^{\frac{1}{p}} \\ & \leq \left(\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_{B(x, 2^{-k}t)}| dy \right)^{\frac{1}{p}} \\ & \quad + |f_{B(x, 2^{-k}t)} - f_{B(x, 2^{-k-1}t)}| + \cdots + |f_{B(x, 2^{-k_0+1}t)} - f_{B(x, 2^{-k_0}t)}| + |f_{B(x, 2^{-k_0}t)} - f_B|. \end{aligned}$$

The first term is bounded by $(2^{-k}t)^\alpha \|f\|_{\mathcal{E}^{p,\alpha}}$, and

$$\begin{aligned} |f_{B(x, 2^{-j}t)} - f_{B(x, 2^{-j-1}t)}| & \leq \frac{|B(x, 2^{-j}t)|}{|B(x, 2^{-j-1}t)|} \frac{1}{|B(x, 2^{-j}t)|} \int_{B(x, 2^{-j-1}t)} |f(y) - f_{B(x, 2^{-j}t)}| dy \\ & \leq c_n 2^n \frac{1}{|B(x, 2^{-j}t)|} \int_{B(x, 2^{-j}t)} |f(y) - f_{B(x, 2^{-j}t)}| dy \\ & \leq c_n 2^n (2^{-j}t)^\alpha \|f\|_{\mathcal{E}^{p,\alpha}} \end{aligned}$$

for $j = k, \dots, k_0$. For the last term, we have

$$\begin{aligned} |f_{B(x, 2^{-k_0}t)} - f_B| & \leq \frac{1}{|B|} \int_B |f(y) - f_{B(x, 2^{-k_0}t)}| dy \\ & \leq \frac{|B(x, 2^{-k_0}t)|}{|B|} \frac{1}{|B(x, 2^{-k_0}t)|} \int_{B(x, 2^{-k_0}t)} |f(y) - f_{B(x, 2^{-k_0}t)}| dy \\ & \leq c_n 4^n (2^{-k_0}t)^\alpha \|f\|_{\mathcal{E}^{p,\alpha}}. \end{aligned}$$

Altogether, we obtain

$$\frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \leq c_n \left(\sum_{j=k}^{k_0} 2^{-j\alpha} \right) \|f\|_{\mathcal{E}^{p,\alpha}}.$$

Since $2r \leq 2^{-k_0}t < 4r$, we have the required estimate. \square

Lemma 2.2. *Let $1 \leq p < \infty$, $B = B(x_0, r)$, $k \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. Then for $x \in B$*

$$\begin{aligned} \left(\frac{1}{|B(x, 2^k r)|} \int_{B(x, 2^k r)} |f(y) - f_B| dy \right)^{\frac{1}{p}} & \leq c_n \left(\sum_{\ell=0}^k 2^{\alpha \ell} \right) r^\alpha \|f\|_{\mathcal{E}^{p,\alpha}(\mathbb{R}^n)} \\ & \leq \begin{cases} c_0 r^\alpha \|f\|_{\mathcal{E}^{p,\alpha}(\mathbb{R}^n)}, & \alpha < 0, \\ c_1 k \|f\|_{\mathcal{E}^{p,\alpha}(\mathbb{R}^n)}, & \alpha = 0, \\ c_2 (2^k r)^\alpha \|f\|_{\mathcal{E}^{p,\alpha}(\mathbb{R}^n)}, & \alpha > 0. \end{cases} \end{aligned}$$

Proof. We see that

$$\begin{aligned} |f_{B(x, 2^{\ell+1}r)} - f_{B(x, 2^\ell r)}| & \leq \frac{|B(x, 2^{\ell+1}r)|^{1+\frac{\alpha}{n}}}{|B(x, 2^\ell r)|} \frac{1}{|B(x, 2^{\ell+1}r)|^{1+\frac{\alpha}{n}}} \int_{B(x, 2^{\ell+1}r)} |f(y) - f_{B(x, 2^{\ell+1}r)}| dy \\ & \leq 2^n c_n (2^{\ell+1}r)^\alpha \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}. \end{aligned}$$

Hence we have

$$\begin{aligned}
& \left(\frac{1}{|B(x, 2^k r)|} \int_{B(x, 2^k r)} |f(y) - f_B| dy \right)^{\frac{1}{p}} \\
& \leq \left(\frac{1}{|B(x, 2^k r)|} \int_{B(x, 2^k r)} |f(y) - f_{B(x, 2^k r)}| dy \right)^{\frac{1}{p}} \\
& \quad + |f_{B(x, 2^k r)} - f_{B(x, 2^{k-1} r)}| + |f_{B(x, 2^{k-1} r)} - f_{B(x, 2^{k-2} r)}| + \cdots + |f_{B(x, 2r)} - f_B| \\
& \leq |B(x, 2^k r)|^{\frac{\alpha}{n}} \|f\|_{\mathcal{E}^{p, \alpha}(\mathbb{R}^n)} + 2^n c_n (2^k r)^\alpha \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)} + \cdots + 2^n c_n r^\alpha \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)} \\
& \leq c \left(\sum_{\ell=0}^k 2^{\alpha \ell} \right) r^\alpha \|f\|_{\mathcal{E}^{p, \alpha}(\mathbb{R}^n)}.
\end{aligned}$$

from which it follows the desired estimates. \square

Lemma 2.3. *Let $B = B(x_0, r)$. Let $x \in B$, $t > 8r$ and $\alpha \in \mathbb{R}$. Then*

$$(2.1) \quad \int_{8r \leq |x-y| < t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy \leq \begin{cases} ctr^\alpha \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha < 0, \\ ct \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha = 0, \\ ct^{1+\alpha} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha > 0. \end{cases}$$

Proof. Let $k_0 \in \mathbb{N}$ satisfy $2r \leq 2^{-k_0} t < 4r$. Then, using Lemma 2.1, we have

$$\begin{aligned}
& \int_{8r \leq |x-y| < t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy \\
& = \sum_{k=0}^{k_0-1} \int_{2^{-k-1} t \leq |x-y| < 2^{-k} t} \frac{|f(y) - f_B|}{|x-y|^{n-1}} dy \\
& \leq \sum_{k=0}^{k_0-1} \frac{1}{(2^{-k-1} t)^{n-1}} \int_{B(x, 2^{-k} t)} |f(y) - f_B| dy \\
& \leq c \sum_{k=0}^{k_0-1} \frac{(2^{-k} t)^n}{(2^{-k-1} t)^{n-1}} \frac{1}{|B(x, 2^{-k} t)|} \int_{B(x, 2^{-k} t)} |f(y) - f_B| dy \\
& \leq c \left(\sum_{k=0}^{k_0-1} 2^{-k} \right) t \frac{1}{|B(x, 2^{-k} t)|} \int_{B(x, 2^{-k} t)} |f(y) - f_B| dy \\
& \leq \begin{cases} ctr^\alpha \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha < 0, \\ ct \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha = 0, \\ ct^{1+\alpha} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha > 0. \end{cases}
\end{aligned}$$

\square

Lemma 2.4. *Let $B = B(x_0, r)$. Let $x \in B$, $t > 8r$, $0 < \beta < 1$ and $\alpha \in \mathbb{R}$.*

$$(2.2) \quad \int_{8r \leq |x-y| < t} \frac{r^\beta |f(y) - f_B|}{|x-y|^{n-1+\beta}} dy \leq \begin{cases} cr^{\beta+\alpha} t^{1-\beta} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha < 0, \\ cr^\beta t^{1-\beta} \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha = 0, \\ cr^\beta t^{1-\beta+\alpha} \|f\|_{\mathcal{E}^{\alpha, 1}(\mathbb{R}^n)}, & \alpha > 0. \end{cases}$$

If $\beta = 1$, we have

$$(2.3) \quad \int_{8r \leq |x-y| < t} \frac{r|f(y) - f_B|}{|x-y|^n} dy \leq \begin{cases} cr^{1+\alpha} \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha < 0, \\ cr \log^2 \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha = 0, \\ crt^\alpha \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha > 0. \end{cases}$$

Proof. Let $k_0 \in \mathbb{N}$ satisfy $2r \leq 2^{-k_0}t < 4r$. Then, using Lemma 2.1, we have

$$\begin{aligned} & \int_{8r \leq |x-y| < t} \frac{r^\beta |f(y) - f_B|}{|x-y|^{n-1+\beta}} dy \\ &= r^\beta \sum_{k=0}^{k_0-1} \int_{2^{-k-1}t \leq |x-y| < 2^{-k}t} \frac{|f(y) - f_B|}{|x-y|^{n-1+\beta}} dy \\ &\leq r^\beta \sum_{k=0}^{k_0-1} \frac{1}{(2^{-k-1}t)^{n-1+\beta}} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \\ &\leq cr^\beta \sum_{k=0}^{k_0-1} \frac{(2^{-k}t)^n}{(2^{-k-1}t)^{n-1+\beta}} \frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \\ &\leq cr^\beta \left(\sum_{k=0}^{k_0-1} 2^{-k(1-\beta)} \right) t^{1-\beta} \frac{1}{|B(x, 2^{-k}t)|} \int_{B(x, 2^{-k}t)} |f(y) - f_B| dy \\ &\leq \begin{cases} cr^{\beta+\alpha} t^{1-\beta} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha < 0, \\ cr^\beta t^{1-\beta} \log \frac{t}{r} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha = 0, \\ cr^\beta t^{1-\beta+\alpha} \|f\|_{\mathcal{E}^{\alpha,1}(\mathbb{R}^n)}, & \alpha > 0. \end{cases} \end{aligned}$$

□

Lemma 2.5. Let $m \in \mathbb{N}$. Let $B = B(x_0, r)$. Let $x \in B$, $0 < \beta \leq 1$, $\alpha \leq 1$, $\gamma < \beta$, and $\alpha + \gamma < \beta$ in the case $0 < \alpha \leq 1$. Suppose furthermore that $\alpha = \alpha_1 + \dots + \alpha_m$ and $\alpha_1, \dots, \alpha_m$ satisfy one of the following three conditions: (i) $\alpha_1, \alpha_2, \dots, \alpha_m < 0$ provided $\alpha < 0$; (ii) $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$ provided $\alpha = 0$; (iii) $\alpha_1, \alpha_2, \dots, \alpha_m > 0$ provided $\alpha > 0$. Then

$$(2.4) \quad \int_{((B(x, 8r)^m)^c} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+\beta-\gamma}} d\vec{y} \leq cr^{\alpha+\gamma} \prod_{i=1}^m \|f_i\|_{\mathcal{E}^{\alpha_i,1}(\mathbb{R}^n)}.$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned}
& \int_{((B(x,8r))^m)^c} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+\beta-\gamma}} d\vec{y} \\
&= \sum_{k=0}^{\infty} \int_{(B(x,2^{k+4}r))^m \setminus (B(x,2^{k+3}r))^m} \frac{r^\beta \prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+\beta-\gamma}} d\vec{y} \\
&\leq cr^\beta \sum_{k=0}^{\infty} \frac{1}{(2^{k+3}r)^{mn+\beta-\gamma}} \int_{(B(x,2^{k+4}r))^m} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| dy_i \\
&\leq cr^\gamma \sum_{k=0}^{\infty} \frac{1}{2^{k(\beta-\gamma)}} \prod_{i=1}^m \frac{1}{|B(x,2^{k+4}r)|} \int_{B(x,2^{k+4}r)} |f_i(y_i) - (f_i)_B| dy_i \\
&\leq \begin{cases} c \sum_{k=0}^{\infty} \frac{1}{2^{k(\beta-\gamma)}} r^{\alpha+\gamma} \prod_{i=1}^m \|f_i\|_{\mathcal{E}^{\alpha_i,1}(\mathbb{R}^n)}, & \alpha < 0, \\ c \sum_{k=0}^{\infty} \frac{(k+4)^m}{2^{k(\beta-\gamma)}} r^\gamma \prod_{i=1}^m \|f_i\|_{\mathcal{E}^{\alpha_i,1}(\mathbb{R}^n)}, & \alpha = 0, \\ c \sum_{k=0}^{\infty} \frac{2^{k\alpha}}{2^{k(\beta-\gamma)}} r^{\alpha+\gamma} \prod_{i=1}^m \|f_i\|_{\mathcal{E}^{\alpha_i,1}(\mathbb{R}^n)}, & \alpha > 0, \end{cases}
\end{aligned}$$

which implies the desired conclusion. \square

3. PROOF OF THEOREM 1.1

Proof. It suffices to verify that for any $f_j \in BMO(\mathbb{R}^n)$, if there exists $y_0 \in \mathbb{R}^n$ such that $\mu(\vec{f})(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$, with $y_0 \in B$,

$$\frac{1}{|B|} \int_B |\mu(\vec{f})(x) - (\mu(\vec{f}))_B| dx \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

For each fixed ball B as above, let r be its radius. Set

$$(3.1) \quad \mu^r(\vec{f})(x) = \left(\int_0^{8r} \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2},$$

and

$$(3.2) \quad \mu^\infty(\vec{f})(x) = \left(\int_{8r}^\infty \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{m(n-1)}} \prod_{i=1}^m f_i(x - y_i) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By the vanishing condition (1.4) of Ω , we can see that for any $x \in B$,

$$\begin{aligned}
\mu^r(\vec{f})(x) &= \mu^r((f_1 - (f_1)_B)\chi_{10B}, \dots, (f_m - (f_m)_B)\chi_{10B})(x) \\
&\leq \mu((f_1 - (f_1)_B)\chi_{10B}, \dots, (f_m - (f_m)_B)\chi_{10B})(x).
\end{aligned}$$

So, by the boundedness of μ in Theorem A, we have

$$\int_B |\mu^r(\vec{f})(x)|^p dx \leq C \prod_{j=1}^m \left(\int_{10B} |f_j(y_j) - (f_j)_B|^{p_j} dy_j \right)^{\frac{p}{p_j}} \leq C |B| \prod_{j=1}^m \|f_j\|_{BMO}^p,$$

where $1 < p_j < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Then

$$\frac{1}{|B|} \int_B |\mu^r(\vec{f})(x)| dx \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

Notice that,

$$\begin{aligned} \frac{1}{|B|} \int_B |\mu(\vec{f})(x) - (\mu(\vec{f}))_B| dx &\leq 2 \frac{1}{|B|} \int_B |\mu^r(\vec{f})(x)| dx \\ &\quad + 2 \frac{1}{|B|} \int_B |\mu^\infty(\vec{f})(x) - \inf_{y \in B} \mu^\infty(\vec{f})(y)| dx. \end{aligned}$$

So we only need to show for any $x \in B$,

$$|\mu^\infty(\vec{f})(x) - \inf_{y \in B} \mu^\infty(\vec{f})(y)| \leq \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)| \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

Thus, to finish the proof of the theorem, it suffice to prove that for any $x, z \in B$,

$$(3.3) \quad |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

It is easy to see that

$$\begin{aligned} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| &= \left| \left(\int_{8r}^\infty |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} - \left(\int_{8r}^\infty |F_t(\vec{f})(z)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right| \\ &\leq \left(\int_{8r}^\infty |F_t(\vec{f})(x) + F_t(\vec{f})(z)| |F_t(\vec{f})(x) - F_t(\vec{f})(z)| \frac{dt}{t} \right)^{\frac{1}{2}}. \end{aligned}$$

Now the vanishing moment of Ω further tells us that for $z \in \mathbb{R}^n$ and $t_1, \dots, t_m > 0$,

$$\begin{aligned} (3.4) \quad &\left| \int_{\prod_{i=1}^m B(z, t_i)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\ &\leq C \sum_{k=-\infty}^{-1} \frac{1}{\prod_{i=1}^m (2^{k+1} t_i)^{n-1}} \int_{\prod_{i=1}^m B(z, 2^{k+1} t_i) \setminus \prod_{i=1}^m B(z, 2^k t_i)} \prod_{j=1}^m |f_j(y_j) - (f_j)_{B(z, 2^{k+1} t_j)}| d\vec{y} \\ &\leq C \sum_{k=-\infty}^{-1} \prod_{i=1}^m 2^k t_i \prod_{j=1}^m \frac{1}{|B(z, 2^{k+1} t_j)|} \int_{B(z, 2^{k+1} t_j)} |f_j(y_j) - (f_j)_{B(z, 2^{k+1} t_j)}| dy_j \\ &\leq C \prod_{j=1}^m t_j \prod_{j=1}^m \|f_j\|_{BMO}. \end{aligned}$$

For $z \in \mathbb{R}^n$, $r > 0$ and $t > 8r$, $B(z, t)^m$ can be decomposed into the following disjoint union

$$B(z, t)^m = (B(z, t) \setminus B(z, 8r))^m \cup (\cup_{i=1}^m B(z, t)^{i-1} \times B(z, 8r) \times B(z, t)^{m-i}) \cup B(z, 8r)^m.$$

So,

$$\begin{aligned}
(3.5) \quad t^m F_t(\vec{f})(z) &= \int_{B(z,t)^m} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&= \int_{(B(z,t) \setminus B(z,8r))^m} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&\quad + \sum_{\ell=1}^m \int_{B(z,t)^{\ell-1} \times B(z,8r) \times B(z,t)^{m-\ell}} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \\
&\quad - (m-1) \int_{B(z,8r)^m} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y}.
\end{aligned}$$

By (3.4) we see that

$$(3.6) \quad t^m |F_t(\vec{f})(z)| \leq C t^m \prod_{j=1}^m \|f_j\|_{BMO},$$

$$\begin{aligned}
(3.7) \quad &\left| \int_{B(z,t)^{\ell-1} \times B(z,8r) \times B(z,t)^{m-\ell}} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\
&\leq C t^{m-1} r \prod_{j=1}^m \|f_j\|_{BMO}, \quad \ell = 1, \dots, m,
\end{aligned}$$

$$(3.8) \quad \left| \int_{B(z,8r)^m} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \leq C r^m \prod_{j=1}^m \|f_j\|_{BMO}.$$

For any fixed $x \in B$ and $t \geq 8r$, set

$$\begin{aligned}
(3.9) \quad H_t(\vec{f})(x, z) &:= \left| \int_{(B(z,t) \setminus B(z,8r))^m} \frac{\Omega(z-y_1, \dots, z-y_m)}{(\sum_{j=1}^m |z-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right. \\
&\quad \left. - \int_{(B(x,t) \setminus B(x,8r))^m} \frac{\Omega(x-y_1, \dots, x-y_m)}{(\sum_{j=1}^m |x-y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right|.
\end{aligned}$$

Then by (3.5), (3.6), (3.7), (3.8) and (3.9) we have for any $x, z \in B$,

$$\begin{aligned}
 (3.10) \quad & |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \\
 & \leq C \left(\int_{8r}^\infty |F_t(\vec{f})(x) - F_t(\vec{f})(z)| \frac{dt}{t} \right)^{1/2} \prod_{j=1}^m \|f_j\|_{BMO}^{\frac{1}{2}} \\
 & \leq C \prod_{j=1}^m \|f_j\|_{BMO}^{\frac{1}{2}} \left(\int_{8r}^\infty (t^{m-1}r + r^m) \frac{dt}{t^{m+1}} \right)^{1/2} \\
 & \quad + C \prod_{j=1}^m \|f_j\|_{BMO}^{\frac{1}{2}} \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \right)^{1/2} \\
 & \leq C \left[\prod_{j=1}^m \|f_j\|_{BMO} + \prod_{j=1}^m \|f_j\|_{BMO}^{\frac{1}{2}} \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \right)^{1/2} \right].
 \end{aligned}$$

Therefore, the proof of inequality (3.3) can be reduced to proving that for any $x, z \in B$,

$$(3.11) \quad \int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

To show this we first introduce some notations. We fix x and z , and for $t > 0$ write

$$\begin{aligned}
 \Xi(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, 8r \leq |z - y| < t\}; \\
 \Xi(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, 8r \leq |x - y| < t\}; \\
 \Gamma(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, |z - y| \geq t\}; \\
 \Gamma(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, |x - y| \geq t\}; \\
 \Lambda(x, t) &= \{y \in \mathbb{R}^n : 8r \leq |x - y| < t, |z - y| < 8r\}; \\
 \Lambda(z, t) &= \{y \in \mathbb{R}^n : 8r \leq |z - y| < t, |x - y| < 8r\}; \\
 \vec{\Theta}(x, t) &= \Theta_1(x, t) \times \cdots \times \Theta_m(x, t), \Theta_i(x, t) \in \{\Xi(x, t), \Gamma(x, t), \Lambda(x, t)\}; \\
 \vec{\Theta}(z, t) &= \Theta_1(z, t) \times \cdots \times \Theta_m(z, t), \Theta_i(z, t) \in \{\Xi(z, t), \Gamma(z, t), \Lambda(z, t)\}.
 \end{aligned}$$

For any y , denote

$$\begin{aligned}
 \Xi(x, y) &= \{t > 0 : 8r \leq |x - y| < t, 8r \leq |z - y| < t\}; \\
 \Xi(z, y) &= \{t > 0 : 8r \leq |z - y| < t, 8r \leq |x - y| < t\}; \\
 \Gamma(x, y) &= \{t > 0 : 8r \leq |x - y| < t, |z - y| \geq t\}; \\
 \Gamma(z, y) &= \{t > 0 : 8r \leq |z - y| < t, |x - y| \geq t\}; \\
 \Lambda(x, y) &= \{t > 0 : 8r \leq |x - y| < t, |z - y| < 8r\}; \\
 \Lambda(z, y) &= \{t > 0 : 8r \leq |z - y| < t, |x - y| < 8r\};
 \end{aligned}$$

$$\Lambda_i(x, y_i) \in \{\Gamma(x, y_i), \Xi(x, y_i)\}, i = 1, \dots, m;$$

$$\Lambda_i(z, y_i) \in \{\Gamma(z, y_i), \Xi(z, y_i)\}, i = 1, \dots, m.$$

Moreover, some immediate consequences are

$$B(x, t) \setminus B(x, 8r) = \Xi(x, t) \cup \Gamma(x, t) \cup \Lambda(x, t),$$

$$B(z, t) \setminus B(z, 8r) = \Xi(z, t) \cup \Gamma(z, t) \cup \Lambda(z, t)$$

and

$$\Xi(x, t) = \Xi(z, t) =: \Xi(t), \quad \Xi(x, y) = \Xi(z, y) =: \Xi(y).$$

Using these notations, we have

$$\begin{aligned}
& H_t(\vec{f})(x, z) \\
&= \left| \int_{(B(x, t) \setminus B(x, 8r))^m} \frac{\Omega(x - y_1, \dots, x - y_m)}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right. \\
&\quad \left. - \int_{(B(z, t) \setminus B(z, 8r))^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\
&\leq \int_{(\Xi(t))^m} \left| \frac{\Omega(x - y_1, \dots, x - y_m)}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} - \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \right| \\
&\quad \times \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \int_{(\Lambda(x, t))^m} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \int_{(\Lambda(z, t))^m} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \int_{\substack{\bar{\Theta}(x, t) \\ \Xi_{\Theta_i}(x, t) = \Gamma(x, t)}} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \int_{\substack{\bar{\Theta}(z, t) \\ \Xi_{\Theta_i}(z, t) = \Gamma(z, t)}} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \sum_{l=1}^{m-1} \int_{(\Xi(x, t))^l} \int_{(\Lambda(x, t))^{m-l}} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\quad + \sum_{l=1}^{m-1} \int_{(\Xi(z, t))^l} \int_{(\Lambda(z, t))^{m-l}} \frac{|\Omega(z - y_1, \dots, z - y_m)|}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&=: \sum_{i=1}^5 H_{t,i}(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,6}^l(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,7}^l(\vec{f})(x, z).
\end{aligned}$$

In the above, we did not explicitly write all the permuted terms for the sake of simplicity.

For $x, z \in B$, we have by Lemma 2.3

$$\begin{aligned}
|H_{t,2}(\vec{f})(x, z)| &\leq C \int_{(B(x,10r) \setminus B(x,8r))^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} d\vec{y} \\
&\leq C \prod_{i=1}^m \int_{8r \leq |x - y_i| \leq 10r} \frac{|f_i(y_i) - (f_i)_B|}{|x - y_i|^{n-1}} dy_i \\
&\leq Cr^m \prod_{j=1}^m \|f_j\|_{BMO},
\end{aligned}$$

which leads to

$$\int_{8r}^{\infty} |H_{t,2}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq C \prod_{j=1}^m \|f_j\|_{BMO},$$

and similarly,

$$\int_{8r}^{\infty} |H_{t,3}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

For $H_{t,4}(\vec{f})$, note that for any $x, z \in B$ the length of t can be controlled by

$$|\cap_{j=1}^m \Theta_j(x, y_j)| \leq |\Theta_i(x, y_i)| = |\Gamma(x, y_i)| \leq ||z - y_i| - |x - y_i|| \leq |z - x| \leq 2r,$$

and for any $t \in \cap_{j=1}^m \Theta_j(x, y_j)$, $t > \frac{1}{m}(\sum_{j=1}^m |x - y_j|)$. Then we can obtain the following estimate by using Lemma 2.5

$$\begin{aligned}
(3.12) \quad &\int_{8r}^{\infty} |H_{t,4}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \\
&\leq C \int_{((B(x,8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \int_{\cap_{j=1}^m \Theta_j(x, y_j)} \frac{dt}{t^{m+1}} d\vec{y} \\
&\leq Cr \int_{((B(x,8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+1}} d\vec{y} \\
&\leq C \prod_{j=1}^m \|f_j\|_{BMO},
\end{aligned}$$

For $H_{t,5}(\vec{f})$, similarly, we have

$$(3.13) \quad \int_{8r}^{\infty} |H_{t,5}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq C \prod_{j=1}^m \|f_j\|_{BMO},$$

For $H_{t,6}^l(\vec{f})$, we may assume $y_1, \dots, y_l \in \Xi(t)$ and $y_{l+1}, \dots, y_m \in \Lambda(x, t)$. Then by the simple calculation and by Lemma 2.3, we have

$$\begin{aligned}
(3.14) \quad & H_{t,6}^l(\vec{f})(x, z) \\
&= \int_{(\Xi(x,t))^l} \int_{(\Lambda(x,t))^{m-l}} \frac{|\Omega(x - y_1, \dots, x - y_m)|}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y} \\
&\leq C \prod_{j=1}^l \int_{8r \leq |x-y_j| < t} \frac{|f_j(y_j) - (f_j)_B|}{|x - y_j|^{n-1}} dy_j \prod_{j=l+1}^m \int_{8r \leq |x-y_j| \leq 10r} \frac{|f_j(y_j) - (f_j)_B|}{|x - y_j|^{n-1}} dy_j \\
&\leq C \left(t \log_2 \frac{t}{r} \right)^l r^{m-l} \prod_{j=1}^m \|f_j\|_{BMO}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
(3.15) \quad & \int_{8r}^{\infty} |H_{t,6}^l(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq C \int_{8r}^{\infty} r^{m-l} \left(t \log_2 \frac{t}{r} \right)^l \frac{dt}{t^{m+1}} \\
&\leq C \int_8^{\infty} \frac{(\log_2 s)^l}{s^{m-l+1}} ds \leq C \prod_{j=1}^m \|f_j\|_{BMO}.
\end{aligned}$$

Similar estimate holds for $H_{t,7}^l(\vec{f})$.

It remains to estimate $H_{t,1}(\vec{f})$. We employ the Lipschitz continuous condition (ii) of Ω and Lemma 2.5 to get the following.

$$\begin{aligned}
(3.16) \quad & \int_{8r}^{\infty} |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \\
&\leq C \int_{((B(x,8r))^c)^m} \frac{|x - z|^\alpha}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)+\alpha}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| \int_{\frac{1}{m}(\sum_{j=1}^m |x-y_j|)}^{\infty} \frac{dt}{t^{m+1}} d\vec{y} \\
&\leq C \int_{((B(x,8r))^c)^m} \frac{r^\alpha}{(\sum_{j=1}^m |x - y_j|)^{mn+\alpha}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| d\vec{y}. \\
&\leq C \prod_{j=1}^m \|f_j\|_{BMO}.
\end{aligned}$$

Thus, we have proved (3.3). This completes the proof of Theorem 1.1. \square

4. PROOFS OF THEOREMS 1.2-1.3

Proof of Theorem 1.2. Similarly to the proof of Theorem 1.1, to prove Theorem 1.2, it suffices to show that for any $f_j \in \mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)$ with $\|f_j\|_{\mathcal{E}^{\alpha_j, p_j}} = 1$, if there exists

$y_0 \in \mathbb{R}^n$ such that $\mu(\vec{f})(y_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni y_0$,

$$\left(\frac{1}{|B|} \int_B |\mu(\vec{f})(x) - \inf_{y \in B} \mu(\vec{f})(y)|^p dx \right)^{1/p} \leq C|B|^{\alpha/n}.$$

Let r be the radius of B , $\mu^r(\vec{f})$ and $\mu^\infty(\vec{f})$ be the same as in (3.1) and (3.2), respectively. Since,

$$\begin{aligned} |\mu(\vec{f})(x) - \inf_{y \in B} \mu(\vec{f})(y)| &\leq \mu(\vec{f})(x) - \inf_{y \in B} \mu^\infty(\vec{f})(y) \\ &\leq |\mu^r(\vec{f})(x)| + \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)|, \end{aligned}$$

by the vanishing moment of Ω , we can write

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |\mu(\vec{f})(x) - \inf_{y \in B} \mu(\vec{f})(y)|^p dx \right)^{1/p} \\ &\leq \left(\frac{1}{|B|} \int_B |\mu^r(\vec{f})(x)|^p dx \right)^{1/p} + \left(\frac{1}{|B|} \int_B \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)|^p dx \right)^{1/p} \\ &\leq \left(\frac{1}{|B|} \int_B |\mu^r((f_1 - (f_1)_{10B})\chi_{10B}, \dots, (f_m - (f_m)_{10B})\chi_{10B})(x)|^p dx \right)^{1/p} \\ &\quad + \left(\frac{1}{|B|} \int_B \sup_{y \in B} |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(y)|^p dx \right)^{1/p} \\ &=: I + II. \end{aligned}$$

By Theorem A, we can write

$$I \leq C \frac{1}{|B|^{1/p}} \left(\prod_{j=1}^m \int_{10B} |f_j(y_j) - (f_j)_{10B}|^{p_j} \right)^{1/p_j} \leq C|B|^{\alpha/n}.$$

Thus, the proof of Theorem 1.2 is now reduced to prove that for any $x, z \in B$,

$$(4.1) \quad |\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \leq Cr^\alpha.$$

If $-m < \alpha < \infty$, $p \in (1, \infty)$ a standard computation gives us that for any $z \in B$ and $t_1, \dots, t_m > 0$,

$$\begin{aligned}
(4.2) \quad & \left| \int_{\prod_{i=1}^m B(z, t_i)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\
& \leq C \sum_{k=-\infty}^0 \frac{1}{\prod_{i=1}^m (2^k t_i)^{n-1}} \int_{\prod_{i=1}^m B(z, 2^k t_i) \setminus \prod_{i=1}^m B(z, 2^{k-1} t_i)} \prod_{j=1}^m |f_j(y_j) - (f_j)_{B(z, 2^k t_j)}| d\vec{y} \\
& \leq C \sum_{k=-\infty}^0 \prod_{i=1}^m (2^k t_i) \left(\frac{1}{\prod_{i=1}^m |B(z, 2^k t_i)|} \int_{\prod_{i=1}^m B(z, 2^k t_i)} \prod_{j=1}^m |f_j(y_j) - (f_j)_{B(z, 2^k t_j)}|^p d\vec{y} \right)^{1/p} \\
& \leq C \prod_{j=1}^m t_j \sum_{k=-\infty}^0 2^{km} \prod_{j=1}^m \left(\frac{1}{|B(z, 2^k t_j)|^m} \int_{(B(z, 2^k t_j))^m} |f_j(y_j) - (f_j)_{B(z, 2^k t_j)}|^{p_j} d\vec{y} \right)^{1/p_j} \\
& \leq C \prod_{j=1}^m t_j \sum_{k=-\infty}^0 2^{km} \prod_{j=1}^m \left(\frac{1}{|B(z, 2^k t_j)|} \int_{(B(z, 2^k t_j))} |f_j(y_j) - (f_j)_{B(z, 2^k t_j)}|^{p_j} dy_j \right)^{1/p_j} \\
& \leq C \prod_{j=1}^m t_j^{1+\alpha_j}.
\end{aligned}$$

(We shall use this fact to prove Theorem 1.3 for $0 < \alpha \leq 1$.)

On the other hand, if $p \in (n, \infty)$ and $\alpha \in (-\infty, 0)$, it follows from Hölder's inequality that

$$\begin{aligned}
(4.3) \quad & \left| \int_{\prod_{i=1}^m B(z, t_i)} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\
& \leq C \left(\int_{\prod_{i=1}^m B(z, t_i)} \prod_{j=1}^m |f_j(y_j) - (f_j)_{B(z, t_j)}|^p d\vec{y} \right)^{1/p} \\
& \quad \times \left(\int_{\prod_{i=1}^m B(z, t_i)} \frac{d\vec{y}}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)p'}} \right)^{1/p'} \\
& \leq C \prod_{j=1}^m t_j^{\frac{n}{p'} - n + 1} \prod_{j=1}^m \left(\int_{\prod_{i=1}^m B(z, t_i)} |f_j(y_j) - (f_j)_{B(z, t_j)}|^{p_j} d\vec{y} \right)^{1/p_j} \\
& \leq C \prod_{j=1}^m t_j^{\frac{n}{p'} - n + 1} \prod_{j=1}^m \left(\prod_{i \neq j} t_i^{n/p_j} t_j^{\alpha_j + n/p_j} \right) = C \prod_{j=1}^m t_j^{\frac{n}{p'} - n + 1} \prod_{j=1}^m t_j^{n/p} t_j^{\alpha_j + n/p_j} \\
& = C \prod_{j=1}^m t_j^{1+\alpha_j}.
\end{aligned}$$

Hence for any $x, z \in B$, according to (4.2) and (4.3), we see that

$$(4.4) \quad t^m |F_t(\vec{f})(z)| \leq C t^{m+\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}},$$

$$(4.5) \quad \left| \int_{B(z, t)^{\ell-1} \times B(z, 8r) \times B(z, t)^{m-\ell}} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \\ \leq C t^{m-1+\alpha-\alpha_\ell} r^{1+\alpha_\ell} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}}, \quad \ell = 1, \dots, m,$$

$$(4.6) \quad \left| \int_{B(z, 8r)^m} \frac{\Omega(z - y_1, \dots, z - y_m)}{(\sum_{j=1}^m |z - y_j|)^{m(n-1)}} \prod_{i=1}^m f_i(y_i) d\vec{y} \right| \leq C r^{m+\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}}.$$

Therefore by (3.5), (4.4), (4.5), (4.6) and (3.9) we have for any $x, z \in B$,

$$|\mu^\infty(\vec{f})(x) - \mu^\infty(\vec{f})(z)| \\ \leq C \left(\int_{8r}^\infty |F_t(\vec{f})(x) - F_t(\vec{f})(z)| \frac{dt}{t^{1-\alpha}} \right)^{\frac{1}{2}} \\ \leq C \left(\int_{8r}^\infty \sum_{j=1}^m t^{m-1+\alpha-\alpha_j} r^{1+\alpha_j} + r^{m+\alpha} \frac{dt}{t^{m+1-\alpha}} \right)^{\frac{1}{2}} + \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \right)^{\frac{1}{2}} \\ \leq C r^\alpha + \left(\int_{8r}^\infty |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \right)^{\frac{1}{2}},$$

where $H_t(\vec{f})(x, z)$ is the same as in the proof of Theorem 1.1. Again decompose $H_t(\vec{f})(x, z)$ into

$$H_t(\vec{f})(x, z) \leq \sum_{i=1}^5 H_{t,i}(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,6}^l(\vec{f})(x, z) + \sum_{l=1}^{m-1} H_{t,7}^l(\vec{f})(x, z).$$

Applying Hölder's inequality and the Lipschitz continuous condition of Ω , we obtain by Lemma 2.5 that for any $x, z \in B$,

$$\int_{8r}^\infty |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \\ \leq C \int_{((B(x, 8r))^c)^m} \frac{|x - z|^\beta}{(\sum_{j=1}^m |x - y_j|)^{m(n-1)+\beta}} \prod_{i=1}^m |f_i(y_i) - (f_i)_B| \int_{\frac{1}{m}(\sum_{j=1}^m |x - y_j|)}^\infty \frac{dt}{t^{m+1-\alpha}} d\vec{y} \\ \leq C \int_{((B(x, 8r))^c)^m} \frac{r^\beta |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+\beta-\alpha}} d\vec{y} \\ \leq C r^{2\alpha}.$$

On the other hand, by Lemma 2.3 we get

$$|H_{t,2}(\vec{f})(x, z)| \leq C \prod_{i=1}^m \int_{8r \leq |x-y_i| \leq 10r} \frac{|f_i(y_i) - (f_i)_B|}{|x - y_i|^{n-1}} dy_i \leq Cr^{m+\alpha}.$$

Hence

$$\int_{8r}^{\infty} |H_{t,2}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq C \int_{8r}^{\infty} \frac{dt}{t^{m+1-\alpha}} r^{m+\alpha} \leq Cr^{2\alpha}.$$

Similarly we get

$$\int_{8r}^{\infty} |H_{t,3}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{2\alpha}.$$

Like as in getting (3.12) in the proof of Theorem 1.1. we obtain by using Lemma 2.5

$$\begin{aligned} \int_{8r}^{\infty} |H_{t,4}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} &\leq Cr \int_{((B(x, 8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{(\sum_{j=1}^m |x - y_j|)^{mn+1-\alpha}} d\vec{y} \\ &\leq Cr^{2\alpha}. \end{aligned}$$

Similar estimate holds for $H_{t,5}(\vec{f})(x, z)$.

As for $H_{t,6}^l(\vec{f})(x, z)$ and $H_{t,7}^l(\vec{f})(x, z)$, similarly by using Lemma 2.3 we have

$$\begin{aligned} &\int_{8r}^{\infty} |H_{t,6}^l(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} + \int_{8r}^{\infty} |H_{t,7}^l(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \\ &\leq Cr^{m-l+\alpha} \int_{8r}^{\infty} t^l \frac{dt}{t^{m+1-\alpha}} \leq Cr^{2\alpha} \end{aligned}$$

Combine the estimates for $H_{t,i}(\vec{f})(x, z)$ ($1 \leq i \leq 5$), $H_{t,6}^l(\vec{f})$ and $H_{t,7}^l(\vec{f})$, we obtain that for any $x, z \in B$, (4.1) holds. We complete the proof of Theorem 1.2. \square

Proof of Theorem 1.3. Using (4.2), we can prove Theorem 1.3 similarly to the proof of Theorem 1.2. The details are omitted. \square

5. EXTENSION TO OPERATORS WITH SEPARATED KERNELS

Definition 5.1. (multilinear Marcinkiewicz integral with separated kernels).

Let $\Omega = \prod_{j=1}^m \Omega_j$ be a function defined on $(\mathbb{R}^n)^m$ with the following properties:

(i) For $j = 1, \dots, m$, Ω_j is homogeneous of degree 0 on \mathbb{R}^n , i.e. for any $\lambda > 0$ and $y \in \mathbb{R}^n$,

$$(5.1) \quad \Omega_j(\lambda y) = \Omega_j(y);$$

(ii) Ω_j is Lipschitz continuous on S^{n-1} , i.e. there is $0 < \alpha < 1$ and $C > 0$ such that for any $\xi, \eta \in \mathbb{R}^n$

$$(5.2) \quad |\Omega_j(\xi) - \Omega_j(\eta)| \leq C|\xi' - \eta'|^\alpha, \quad j = 1, \dots, m.$$

where $y' = \frac{y}{|y|}$;

(iii) The integration of Ω_j on each unit sphere vanishes,

$$(5.3) \quad \int_{S^{n-1}} \Omega_j(y) dy = 0, \quad j = 1, \dots, m.$$

For any $\vec{f} = (f_1, \dots, f_m) \in S \times \dots \times S$, we define the operator F_t for any $t > 0$ as

$$\begin{aligned}
 (5.4) \quad F_t(\vec{f})(x) &= \frac{\chi_{(B(0,t))^m} \Omega(\cdot)}{t^m \prod_{j=1}^m |\cdot|^{n-1}} * (f_1 \otimes \dots \otimes f_m)(x) \\
 &= \frac{1}{t^m} \int_{(B(0,t))^m} \prod_{j=1}^m \frac{\Omega_j(y_j)}{|y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y}, \\
 &= \frac{1}{t^m} \prod_{j=1}^m \int_{B(0,t)} \frac{\Omega_j(y_j)}{|y_j|^{n-1}} f_j(x - y_j) dy_j,
 \end{aligned}$$

Finally, the multilinear Marcinkiewicz integral $\tilde{\mu}$ is defined by

$$(5.5) \quad \tilde{\mu}(\vec{f})(x) = \left(\int_0^\infty |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

It is easily seen that

$$\tilde{\mu}(\vec{f})(x) \leq C \left(\int_0^\infty \left| \frac{1}{t} \int_{B(0,t)} \frac{\Omega_1(y)}{|y|^{n-1}} f_1(x - y_1) dy \right|^2 \frac{dt}{t} \right)^{1/2} \prod_{j=2}^m M f_j(x).$$

Hence, the $L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \longrightarrow L^p$ boundedness follows easily for $1 < p_1, p_2, \dots, p_m < \infty$ and $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$. For $\tilde{\mu}$, we have

Theorem 5.1. *Theorems 1.1 and 1.2 are still true for the operator $\tilde{\mu}$. Theorem 1.3 also holds under the additional condition $0 < \alpha_j < 1/m$, $j = 1, \dots, m$.*

5.1. Proof of Theorem 5.1. We only need to show that: For any $f_j \in \mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)$, if there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{\mu}(\vec{f})(x_0) < \infty$, then for any ball $B \subset \mathbb{R}^n$ with $B \ni x_0$,

$$(5.6) \quad |\tilde{\mu}^\infty(\vec{f})(x_0) - \tilde{\mu}^\infty(\vec{f})(x)| \leq C r^\alpha \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)}.$$

In fact, if the above inequality (5.6) holds, then

$$\tilde{\mu}^\infty(\vec{f})(x) \leq \tilde{\mu}^\infty(\vec{f})(x_0) + C r^\alpha \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)} < +\infty.$$

Because of the $L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \longrightarrow L^p$ boundedness, we have

$$\left(\frac{1}{|B|} \int_B |\tilde{\mu}^r(\vec{f})(x)|^p dx \right)^{1/p} \leq |B|^\alpha \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, p_j}(\mathbb{R}^n)} < +\infty.$$

This implies that $\tilde{\mu}^r(\vec{f})(x) < +\infty$ a.e. on B . Therefore, $\tilde{\mu}(\vec{f})(x) < +\infty$ a.e. on B .

Since $B \ni x_0$ is arbitrary, it follows that $\tilde{\mu}(\vec{f})(x) < +\infty$ for almost every on $x \in \mathbb{R}^n$. To show inequality (5.6), we follow the steps in the proof of Theorems 1.1-1.3. As before, we see that

$$\left| \int_{B(0,t)} \frac{\Omega_j(y_j)}{|y_j|^{n-1}} f_j(x - y_j) dy_j \right| \leq C t^{1+\alpha_j} \|f_j\|_{\mathcal{E}^{\alpha_j, 1}}.$$

From this we see easily that the same estimates as 4.4, 4.5, and 4.6 hold. Hence, to show inequality (5.6) we have only to show

$$\int_{8r}^{\infty} |H_t(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{2\alpha}.$$

where as before $H_t(\vec{f})(x, z)$ is defined by

$$\begin{aligned} H_t(\vec{f})(x, z) := & \left| \int_{(B(z,t) \setminus B(z,8r))^m} \prod_{j=1}^m \frac{\Omega_j(z - y_1)}{|z - y_j|^{n-1}} f_j(y_j) d\vec{y} \right. \\ & \left. - \int_{(B(x,t) \setminus B(x,8r))^m} \prod_{j=1}^m \frac{\Omega_j(x - y_1)}{|x - y_j|^{n-1}} f_j(y_j) d\vec{y} \right|. \end{aligned}$$

We may use the following estimate as before:

$$H_t(\vec{f})(x, z) \leq \sum_{i=1}^5 H_{t,i}(\vec{f})(x, z) + \sum_{\ell=1}^{m-1} H_{t,6}^{\ell}(\vec{f})(x, z) + \sum_{\ell=1}^{m-1} H_{t,7}^{\ell}(\vec{f})(x, z).$$

(I) As for $H_{t,1}(\vec{f})(x, z)$, we get

$$\begin{aligned} |H_{t,1}(\vec{f})(x, z)| & \leq C \int_{\Xi(x,t)^m} \sum_{i=1}^m \frac{r^{\beta}}{|x - y_i|^{n-1+\beta}} \prod_{\ell \neq i} \frac{1}{|x - y_{\ell}|^{n-1}} \prod_{j=1}^m |f_j(y) - (f_j)_B| d\vec{y} \\ & \leq C \sum_{j=1}^m \int_{8r \leq |x - y_j| < t} \frac{r^{\beta} |f_j(y) - (f_j)_B|}{|x - y_j|^{n-1+\beta}} dy_j \prod_{\ell \neq j} \int_{8r \leq |x - y_{\ell}| < t} \frac{|f_{\ell}(y) - (f_{\ell})_B|}{|x - y_{\ell}|^{n-1}} dy_{\ell}. \end{aligned}$$

(i) In the case $\alpha < 0$, using Lemmas 2.4 and 2.3, we have

$$|H_{t,1}(\vec{f})(x, z)| \leq C \sum_{j=1}^m r^{\beta+\alpha_j} t^{1-\beta} \prod_{\ell \neq j} (tr^{\alpha_{\ell}}) \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}} = Cr^{\alpha+\beta} t^{m-\beta} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}.$$

So, we have

$$\int_{8r}^{\infty} |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{\beta+\alpha} \int_{8r}^{\infty} \frac{dt}{t^{1+\beta-\alpha}} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}} \leq Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}.$$

(ii) In the case $\alpha = 0$, using Lemmas 2.4 and 2.3, we have

$$|H_{t,1}(\vec{f})(x, z)| \leq Cr^{\beta} t^{1-\beta} \left(\log \frac{t}{r} \right) \left(t \log \frac{t}{r} \right)^{m-1} \prod_{j=1}^m \|f_j\|_{BMO} = Cr^{\beta-\gamma} t^{m+\gamma-\beta} \prod_{j=1}^m \|f_j\|_{BMO},$$

for some $0 < \gamma < \beta$. So, we have

$$\int_{8r}^{\infty} |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{\beta-\gamma} \int_{8r}^{\infty} \frac{dt}{t^{1+\beta-\gamma}} \prod_{j=1}^m \|f_j\|_{BMO} \leq C \prod_{j=1}^m \|f_j\|_{BMO}.$$

(iii) In the case $0 < \alpha < \beta/2$, using Lemmas 2.4 and 2.3, we have

$$|H_{t,1}(\vec{f})(x, z)| \leq Cr^\beta t^{1-\beta+\alpha_j} \prod_{\ell \neq j} t^{1+\alpha_j} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}} = Cr^\beta t^{m+\alpha-\beta} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}.$$

So, we have

$$\int_{8r}^\infty |H_{t,1}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^\beta \int_{8r}^\infty \frac{dt}{t^{1+\beta-2\alpha}} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}} \leq Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}.$$

(II) As for $H_{t,2}(\vec{f})$, we have for $x, z \in B$,

$$\begin{aligned} |H_{t,2}(\vec{f})(x, z)| &\leq C \int_{(B(x,10r) \setminus B(x,8r))^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{\prod_{i=1}^m |z - y_i|^{n-1}} d\vec{y} \\ &\leq C \prod_{i=1}^m \frac{1}{r^{n-1}} \int_{B(x,10r)} |f_i(y_i) - (f_i)_B| dy_i \\ &\leq Cr^{m+\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}, \end{aligned}$$

which leads to

$$\int_{8r}^\infty |H_{t,2}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}},$$

if $\alpha < m$. Similarly,

$$\int_{8r}^\infty |H_{t,3}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}.$$

(III) For $H_{t,4}(\vec{f})$, we have for any $x, z \in B$

$$|\cap_{j=1}^m \Theta_j(x, y_j)| \leq |\Theta_i(x, y_i)| = |\Gamma(x, y_i)| \leq ||z - y_i| - |x - y_i|| \leq |z - x| \leq 2r,$$

and for any $t \in \cap_{j=1}^m \Theta_j(x, y_j)$, $t > \frac{1}{m}(\sum_{j=1}^m |x - y_j|)$. So, we obtain the following estimate:

$$\begin{aligned} (5.7) \quad &\int_{8r}^\infty |H_{t,4}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \\ &\leq C \int_{((B(x,8r))^c)^m} \frac{\prod_{i=1}^m |f_i(y_i) - (f_i)_B|}{\prod_{i=1}^m |x - y_i|^{n-1}} \int_{\cap_{j=1}^m \Theta_j(x, y_j)} \frac{dt}{t^{m+1-\alpha}} d\vec{y} \\ &\leq C \int_{((B(x,8r))^c)^m} \prod_{i=1}^m \frac{|f_i(y_i) - (f_i)_B|}{|x - y_i|^{n-1}} \frac{r}{(\sum_{j=1}^m |x - y_j|)^{m+1-\alpha}} d\vec{y}. \\ &\leq C \prod_{i=1}^m \int_{(B(x,8r))^c} \frac{r^{1/m} |f_i(y_i) - (f_i)_B|}{|x - y_i|^{n+1/m-\alpha_i}} dy_i. \end{aligned}$$

We have a similar estimate for $H_{t,5}(\vec{f})(x, z)$. Hence, using Lemma 2.5, we get

$$\begin{aligned} & \int_{8r}^{\infty} |H_{t,4}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} + \int_{8r}^{\infty} |H_{t,5}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \\ & \leq C \prod_{j=1}^m \int_{(B(x, 8r))^c} \frac{r^{1/m} |f_j(y_j) - (f_j)_B|}{|x - y_j|^{n+1/m-\alpha_j}} dy_j + C \prod_{j=1}^m \int_{(B(z, 8r))^c} \frac{r^{1/m} |f_j(y_j) - (f_j)_B|}{|z - y_j|^{n+1/m-\alpha_j}} dy_j \\ & \leq Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}}, \end{aligned}$$

if $\alpha_j < 1/m$, $j = 1, \dots, m$.

(IV) As for $H_{t,6}(\vec{f})$, we see that

$$|H_{t,6}^{\ell}(\vec{f})(x, z)| \leq \prod_{j=1}^{\ell} \int_{8r \leq |x-y_j| < t} \frac{|f_j(y_j) - (f_j)_B|}{|x - y_j|^{n-1}} dy_j \prod_{j=\ell+1}^m \int_{8r \leq |x-y_j| < 10r} \frac{|f_j(y_j) - (f_j)_B|}{|x - y_j|^{n-1}} dy_j.$$

(i) In the case $\alpha < 0$, we have by using Lemma 2.3

$$|H_{t,6}^{\ell}(\vec{f})(x, z)| \leq c \prod_{j=1}^{\ell} (tr^{\alpha_j}) \prod_{j=1}^{\ell} \|f_j\|_{\mathcal{E}^{\alpha_j, 1}} \prod_{j=\ell+1}^m r^{1+\alpha_j} \prod_{j=\ell+1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}} = ct^{\ell} r^{m-\ell+\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}}.$$

Hence we have

$$\int_{8r}^{\infty} |H_{t,6}^{\ell}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} \leq Cr^{m-\ell+\alpha} \int_{8r}^{\infty} \frac{dt}{t^{m-\ell+1-\alpha}} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}} = Cr^{2\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}}.$$

We have a similar estimate for $H_{t,7}^{\ell}(\vec{f})$.

(ii) In the case $\alpha = 0$, we have by using Lemma 2.3

$$|H_{t,6}^{\ell}(\vec{f})(x, z)| \leq c \left(t \log \frac{t}{r} \right)^{\ell} \prod_{j=1}^{\ell} \|f_j\|_{BMO} r^{m-\ell} \prod_{j=\ell+1}^m \|f_j\|_{BMO} \leq ct^{(1+\gamma)\ell} r^{m-(1+\gamma)\ell} \prod_{j=1}^m \|f_j\|_{BMO},$$

for some sufficiently small $\gamma > 0$. Hence we have

$$\int_{8r}^{\infty} |H_{t,6}^{\ell}(\vec{f})(x, z)| \frac{dt}{t^{m+1}} \leq Cr^{m-(1+\gamma)\ell} \int_{8r}^{\infty} \frac{dt}{t^{m-(1+\gamma)\ell+1}} \prod_{j=1}^m \|f_j\|_{BMO} = C \prod_{j=1}^m \|f_j\|_{BMO}.$$

(iii) In the case $\alpha > 0$, we have by using Lemma 2.3

$$|H_{t,6}^{\ell}(\vec{f})(x, z)| \leq c \prod_{j=1}^{\ell} t^{1+\alpha_j} \|f_j\|_{\mathcal{E}^{\alpha_j, 1}} \prod_{j=\ell+1}^m r^{1+\alpha_j} \|f_j\|_{\mathcal{E}^{\alpha_j, 1}} = ct^{\ell+\sum_{j=1}^{\ell} \alpha_j} r^{m-\ell+\sum_{j=\ell+1}^m \alpha_j} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j, 1}}.$$

Hence we have

$$\begin{aligned} \int_{8r}^{\infty} |H_{t,6}^{\ell}(\vec{f})(x, z)| \frac{dt}{t^{m+1-\alpha}} &\leq Cr^{m-\ell+\sum_{j=\ell+1}^m \alpha_j} \int_{8r}^{\infty} \frac{dt}{t^{m-(\ell+\sum_{j=1}^{\ell} \alpha_j)+1-\alpha}} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}} \\ &= Cr^{2m\alpha} \prod_{j=1}^m \|f_j\|_{\mathcal{E}^{\alpha_j,1}}, \end{aligned}$$

if $\alpha - \alpha_j < 1$, $j = 1, \dots, m$, in particular, if $\alpha_j < 1/m$, $j = 1, \dots, m$.

We have a similar estimate for $H_{t,7}^{\ell}(\vec{f})$.

This completes the proof of Theorem 5.1. \square

5.2. More general type kernels.

Definition 5.2. (Multilinear Marcinkiewicz integral) Let Ω be a function defined on $(\mathbb{R}^n)^m$ satisfying the conditions (i), (ii) and (iii) in Definition 5.2. For any $\vec{f} = (f_1, \dots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \dots \times \mathcal{S}(\mathbb{R}^n)$, we define the operator F_t for any $t > 0$ as

$$(5.8) \quad F_t(\vec{f})(x) = \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(\vec{y})}{\prod_{i=1}^m |y_i|^{n-1}} \prod_{i=1}^m f_i(x - y_i) d\vec{y},$$

where $|\vec{y}| = |y_1| + \dots + |y_m|$ and $B(x, t) = \{y \in \mathbb{R}^n : |y - x| \leq t\}$. Define the multilinear Marcinkiewicz integral μ by

$$(5.9) \quad \mu(\vec{f})(x) = \left(\int_0^{\infty} |F_t(\vec{f})(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Then an open conjecture is as follows:

Conjecture 5.2. *If Ω satisfies the conditions (i), (ii) and (iii) in Definition 5.2 and μ is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times \dots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $1 < q_1, q_2, \dots, q_m < \infty$ with $1/q = 1/q_1 + 1/q_2 + \dots + 1/q_m$, then, μ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $1 < p_1, p_2, \dots, p_m < \infty$ with $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$.*

We can only prove it for some special case, here we give two examples.

Example 5.3.

$$\Omega(x) = \prod_{j=1}^m \Omega_j(x_j),$$

where Ω_j is homogeneous of degree 0 on \mathbb{R}^n , Lipschitz continuous on S^{n-1} , and $\int_{S^{n-1}} \Omega_j(y_j) dy_j = 0$ for $j = 1, \dots, m$.

Example 5.4.

$$\Omega(x) = \sin\left(\prod_{j=1}^m \Omega_j(x_j)\right),$$

where Ω_j is odd and homogeneous of degree 0 on \mathbb{R}^n , and Lipschitz continuous on S^{n-1} for $j = 1, \dots, m$.

Claim 1. For $1 < p_1, p_2, \dots, p_m < \infty$ with $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$, μ in Example 5.4 is bounde from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. Since $\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$ for $|z| < \infty$, we have

$$(5.10) \quad \sin\left(\prod_{j=1}^m \Omega_j(x_j)\right) = \sum_{k=0}^{\infty} (-1)^k \frac{(\prod_{j=1}^m \Omega_j(x_j))^{2k+1}}{(2k+1)!}.$$

Since the above convergence of $\sin z$ is uniform on every compact set of \mathbb{C} , and Ω_j 's are Lipschitz continuous on S^{n-1} , this convergence is uniform on $B(0, t) \times B(0, t) \times \dots \times B(0, t)$ for any fixed $0 \leq t < \infty$. Hence, for any $f_j \in \mathcal{S}(\mathbb{R}^n)$ ($j = 1, \dots, m$), we have

$$(5.11) \quad \begin{aligned} & \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-1)^k}{(2k+1)!} \int_{(B(0,t))^m} \frac{(\prod_{j=1}^m \Omega_j(y_j))^{2k+1}}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y} \\ &= \int_{(B(0,t))^m} \frac{\Omega(y_1, y_2, \dots, y_m)}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y}. \end{aligned}$$

So, by Fatou's Lemma we get

$$(5.12) \quad \begin{aligned} & \mu(\vec{f})(x) \\ &= \left(\int_0^\infty \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{\Omega(y_1, y_2, \dots, y_m)}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \lim_{N \rightarrow \infty} \left| \frac{1}{t^m} \sum_{k=0}^N \frac{(-1)^k}{(2k+1)!} \int_{(B(0,t))^m} \frac{(\prod_{j=1}^m \Omega_j(y_j))^{2k+1}}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \liminf_{N \rightarrow \infty} \left(\int_0^\infty \left| \frac{1}{t^m} \sum_{k=0}^N \frac{(-1)^k}{(2k+1)!} \int_{(B(0,t))^m} \frac{(\prod_{j=1}^m \Omega_j(y_j))^{2k+1}}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2} \\ &\leq \liminf_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-1)^k}{(2k+1)!} \left(\int_0^\infty \left| \frac{1}{t^m} \int_{(B(0,t))^m} \frac{(\prod_{j=1}^m \Omega_j(y_j))^{2k+1}}{\prod_{j=1}^m |y_j|^{n-1}} \prod_{j=1}^m f_j(x - y_j) d\vec{y} \right|^2 \frac{dt}{t} \right)^{1/2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\int_0^\infty \left| \prod_{j=1}^m \frac{1}{t} \int_{B(0,t)} \frac{(\Omega_j(y_j))^{2k+1}}{|y_j|^{n-1}} f_j(x - y_j) dy_j \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

We see that

$$\left| \frac{1}{t} \int_{B(0,t)} \frac{(\Omega_j(y_j))^{2k+1}}{|y_j|^{n-1}} f_j(x - y_j) dy_j \right| \leq C \|\Omega\|_\infty^{2k+1} M f_j(x).$$

So, we have

$$\begin{aligned}
 (5.13) \quad & \left(\int_0^\infty \left| \prod_{j=1}^m \frac{1}{t} \int_{B(0,t)} \frac{(\Omega_j(y_j))^{2k+1}}{|y_j|^{n-1}} f_j(x - y_j) dy_j \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & \leq C \|\Omega\|_\infty^{2k+1} \prod_{j=2}^m M f_j(x) \left(\int_0^\infty \left| \frac{1}{t} \int_{B(0,t)} \frac{(\Omega_1(y_1))^{2k+1}}{|y_1|^{n-1}} f_1(x - y_1) dy_1 \right|^2 \frac{dt}{t} \right)^{1/2},
 \end{aligned}$$

We see that

$$\|(\Omega_1(y_1))^{2k+1}\|_{\text{Lip}_\alpha(S^{n-1})} \leq Ck \|\Omega_1\|_\infty^{2k+1} \|\Omega_1\|_{\text{Lip}_\alpha(S^{n-1})},$$

which implies that for every $1 < q < \infty$ there exist some $A > 0$ and $C_q > 0$ such that

$$\left\| \left(\int_0^\infty \left| \frac{1}{t} \int_{B(0,t)} \frac{(\Omega_1(y_1))^{2k+1}}{|y_1|^{n-1}} f_1(x - y_1) dy_1 \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^q(\mathbb{R}^n)} \leq C_q k A^k \|f_1\|_{L^q(\mathbb{R}^n)}.$$

By (5.12), (5.13), (5.14), the L^p boundedness of the Hardy-Littlewood maximal function, Minkowski's inequality and Young's inequality, we obtain for $1 < p_1, p_2, \dots, p_m < \infty$ with $1/p = 1/p_1 + 1/p_2 + \dots + 1/p_m$

$$(5.15) \quad \|\mu(\vec{f})\|_{L^p(\mathbb{R}^n)} \leq C \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} k A^k \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

which completes the proof of our Claim 1. □

REFERENCES

- [1] A. Benedek, A.P. Calderón and R. Panzone, *Convolution operators on Banach value functions*, Proc Nat Acad Sci, 48 (1962), 256-365.
- [2] J. Chen, D. Fan and Y. Ying, *Singular integral operators on function spaces*, J. Math. Anal. Appl. **276** (2002), 691-708.
- [3] J. Chen and C. Zhang, *Boundedness of g -functions on Triebel-Lizorkin spaces*, Taiwanese J Math, **13** (2009), 973-981.
- [4] X. Chen, Q. Xue and K. Yabuta, *On multilinear Littlewood-Paley operators*, Nonlinear Analysis: TMA., **115** (2015), 25-40.
- [5] M. Christ and J. L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math., **159** (1987), 51-80.
- [6] R. R. Coifman, A. McIntosh and Y. Meyer, *L 'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. Math., **116** (1982), 361-387.
- [7] R. R. Coifman, Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc., **212** (1975), 315-331.
- [8] R. R. Coifman, Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble), **28** (1978), 177-202.
- [9] R. R. Coifman, Y. Meyer, *Au-delà des opérateurs pseudo-différentiels*, Asterisque, **57** (1978), 1-185.
- [10] Y. Ding, D. Fan and Y. Pan, *L^p -boundedness of Marcinkiewicz integrals with Hardy space function kernels*, Acta Mathematica Sinica (Series B), 16, (2000), 593-600.
- [11] Y. Ding, D. Fan and Y. Pan, *On the L^p boundedness of Marcinkiewicz integrals*, Michigan Math. J., 50 (2002), 17-26.

- [12] Y. Ding, C.-C. Lin and S. Shao, *On Marcinkiewicz integral with variable kernels*, Indiana Univ. Math. J., **53** (2004), 805-822.
- [13] Y. Ding, S. Lu and Q. Xue, *On Marcinkiewicz integral with homogeneous kernels*, J. Math. Anal. Appl., **245**, 471-488 (2000).
- [14] Y. Ding, S. Lu and Q. Xue, *Marcinkiewicz Integral on Hardy spaces*, Integr. Equ. Oper. Theory., **42** (2002), no. 2, 174-182.
- [15] L. Grafakos, R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. Math., **165** (2002), 124-164.
- [16] L. Grafakos and R. H. Torres, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J., **51** (5) (2002), 1261-1276.
- [17] Y. Han, *Some properties of S-function and Marcinkiewicz integrals*, Acta Sci. Natur. Univ. Pekingniss **5** (1987), 21-34.
- [18] S. He, Q. Xue, T. Mei and K. Yabuta, *Existence and boundedness of multilinear Littlewood-Paley operators on Campanato spaces*, J. Math. Anal. Appl., **432** (2015), 86-102.
- [19] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math., **104** (1960), 93-140.
- [20] S. Janson, M. H. Taibleson and G. Weiss, *Elementary characterizations of the Morrey-Campanato spaces*, in: Harmonic analysis (Cortona, 1982), Lecture Notes in Mathematics No. 992 (Springer-Verlag, 1983), 101-114.
- [21] C. E. Kenig, E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett., **6** (1999), 1-15.
- [22] D. S. Kurtz, *Littlewood-Paley operators on BMO*, Proc. Amer. Math. Soc., **99** (1987), 657-666.
- [23] A. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math., **220** (2009), 1222-1264.
- [24] S. Z. Lu, *Marcinkiewicz integrals with rough kernels*, Front. Math. China **3** (2008), 1-14.
- [25] S. Qiu, *Boundedness of Littlewood-Paley operators and Marcinkiewicz integral on $\mathcal{E}^{\alpha,p}$* , J. Math. Res. Exposition, **12** (1992), 41-50.
- [26] M. Sakamoto and K. Yabuta, *Boundedness of Marcinkiewicz functions*, Studia Math., **135** (1999), 103-142.
- [27] E. M. Stein, *On the function of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430-466.
- [28] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), No. 2, 430-466.
- [29] S. Wang, *Some properties of g-functions*, Sci. Sinica., Series A, **28** (1985), 252-262.
- [30] S. Wang, *Boundedness of the Littlewood-Paley g-functions on $Lip_\alpha(\mathbb{R}^n)$ ($0 < \alpha < 1$)*, Illinois. J. Math., **33** (1989), 531-541.
- [31] S. Wang, J. Chen, *Some notes on square function operators*, Chinese Ann. Math., Series A, **11** (1990), 630-638.
- [32] S. Sato, K. Yabuta, *Multilinearized Littlewood-Paley operators*, Sci. Math. Jpn., **55** (3) (2002), 447-453.
- [33] Z. Si, L. Wang and Y. Jiang, *Fractional type Marcinkiewicz integral on Hardy spaces*, J. Math. Res. Exposition, **31** (2011), 233-241.
- [34] Y. Sun, *On the existence and boundedness of square function operators on Campanato spaces*, Nagoya Math. J., Vol. **173** (2004), 139-151.
- [35] A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math. **60/61** (1990), 235-243.
- [36] Q. Xue, K. Yabuta, J. Yan, *Fractional type Marcinkiewicz integral operators on function spaces*, Forum Math. **27** (2015), no. 5, 3079-3109.
- [37] K. Yabuta, *Boundedness of Littlewood-Paley operators*, Math. Japon., **43** (1996), 143-150.
- [38] K. Yabuta, *A multilinearization of Littlewood-Paley's g-function and Carleson measures*, Tohoku Math. J., **34** (1982), 251-275.

- [39] K. Yabuta, *Existence and boundedness of g_λ^* -function and Marcinkiewicz functions on Campanato spaces*, Sci. Math. Jpn., **9**, (2003), 59-78.

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